Subgroups of a Subnormal Subgroup in Division Rings

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Abstract

Let D be a division ring with center F and the multiplicative group D^*. Our main purpose in this paper is to investigate the subgroup structure of an arbitrary subnormal subgroup G of D^*. In particular, we prove that if D is locally finite, then G contains a noncyclic free subgroup. The structure of maximal subgroups of G is also investigated. The new obtained results carry over some previous results about maximal subgroups of D^* to maximal subgroups of an arbitrary subnormal subgroup G of D^*.

Key words: division ring, maximal subgroup, subnormal subgroup, noncyclic free subgroup, FC-element.
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1 Introduction

Let D be a division ring with center F and D^* be the multiplicative group of D. The subgroup structure of D^* is one of subjects which attract the attention of many authors for last time. A particular interest is that to study maximal subgroups of D^*, see for example, [1], [2], [4], [8], [14], [21]. In this paper we replace D^* by its arbitrary subnormal subgroup G and we study the subgroup structure of G. Recall that in [1], [4], [21], Akbari et al. and Mahdavi-Hezavehi study maximal subgroups of D^* and many nice properties of such subgroups were obtained. In the present paper, studying maximal subgroups of G, we get in many cases the similar results for these subgroups as the results obtained in [1], [4], [21] ... for maximal subgroups of D^*. Other problem we study is the problem of the existence of noncyclic free groups in a subnormal subgroups of a division ring. This problem plays an important role for the understanding of the structure of division rings. For more information we refer to the works [5]-[7], [16]-[18], and [25]. Section 4 is devoted to this problem and the new result we get in Theorem 4.4 gives the affirmative answer to Conjecture 2 from [7] for the case of locally finite division rings.

In Section 2, we give some generalization of the well-known Wedderburn's Factorization theorem about polynomials over division rings. This generalization and its corollaries will be used as important tools for our study in the next sections of the paper. In the last part of this section, as

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some illustration, we use them to generalize some previous results by Mahdavi-Hezavehi et al. in [19].

Section 3 is devoted to the study of maximal subgroups of a subnormal subgroup G of D^*. Firstly, we consider the case when a maximal subgroup M of G contains some non-central FC-element. The properties obtained in Theorem 3.5 are principal tools for our study in the next parts of the paper. Also, in Theorem 3.7, we give a perfect description of the structure of D when M is metabelian. In Section 5, we concentrate our study to the particular case when M is a maximal subgroup of D^*. The results we get are generalizations of the result by Akbari et al. in [1, Theorem 6] and the result by B. X. Hai and N. V. Thin in [8, Theorem 3.2] . More exactly, we replace the condition of algebraicity of M in [1, Theorem 6] and the condition of algebraicity of D in [8, Theorem 3.2] by the weaker condition of algebraicity of derived subgroup M' and the obtained results are the same. We refer to [2], [10], and [14] for more information about the existence of maximal subgroups in division rings.

Throughout this paper, for a ring R, the symbol R^* denotes the group of all units in R. For a non-empty subset S of D, we denote by $F[S]$ and $F(S)$ the subring and the division subring of D respectively, generated by the set $F \cup S$, where $F = Z(D)$ is the center of D. We say that a subgroup G of D^* is *absolutely irreducible* if $F[G] = D$. Given a group G, a subset S and a subgroup H of G, we denote by $Z(G)$, G', $C_G(S)$ and $H_G = \bigcap_{x\in G} xHx^{-1}$ the center of G, the derived group of G, the centralizer of S in G and the core of H in G respectively. If $x \in G$ and H is a subgroup of G, then we denote by x^H the set of all elements $x^h = hxh^{-1}, h \in H$. If x^G is finite, then we say that x is an *FC-element* of G. The set of all FC-elements of G is called the *FC-center* of G. If $x, y \in G$ then $[x, y] = xyx^{-1}y^{-1}$ and $[H, K]$ is the subgroup of G generated by all elements $[h, k], h \in H,\ k \in K$. We say that an element $x \in D$ is *radical* over F if there exists a positive integer $n(x)$ depending on x such that $x^{n(x)} \in F$. A non-empty subset S of D is *radical* over F if every element of S is radical over F. We write $H \leq G$ and $H < G$ if H is a subgroup and proper subgroup of G respectively. All other notation and symbols in this paper are standard and one can find, for example, in [15], [22], [23], [26], [27].

2 Algebraicity over a division subring and the generalization of Wedderburn's Factorization theorem

Let D be a division ring with center F, and let A be a conjugacy class of D which is algebraic over F with minimal polynomial $f(t) \in F[t]$ of degree n. Then, there exist $a_1, \dots, a_n \in A$ such that

$$f(t) = (t - a_1) \dots (t - a_n) \in D[t].$$

This factorization theorem due to Wedderburn (see [15, (16.9), p. 265]) plays an important role in the theory of polynomials over a division ring and its applications in different problems are well-known. In this section, firstly we consider the similar question in more general circumstance in order to get the analogous theorem which could be used to generalize some previous results of other authors about subgroups in division rings.

Let $K \subseteq D$ be division rings and $\alpha \in D$. We say that α is *(right) algebraic* over K if there exists some nonzero polynomial $f(t) \in K[t]$ having α as a right root. A monic polynomial from $K[t]$ with smallest degree having α as a right root is called a *minimal polynomial* of α over K. Throughout this paper we consider only right roots and right algebraicity, so we shall always omit the prefix "right". The minimal polynomial of α over K is unique, but it may not be irreducible as the following example shows:

Let $\mathbb{H}$ be the division ring of real quaternions. Then, $f(t) = t^2 + 1 \in \mathbb{C}[t]$ is the minimal polynomial of j and k over $\mathbb{C}$. Here, $\{1, i, j, k\}$ is standard basis of $\mathbb{H}$ over $\mathbb{R}$.

The proof of the following lemma is a simple modification of the proof of Lemma (16.5) in [15], so we now omit it.

Lemma 2.1. *Let R be a ring, D a division subring of R and suppose M is a subgroup of R^* normalizing D^*. If $K = C_D(M)$ and $x \in D^*$ is algebraic over K with the minimal polynomial $f(t) \in K[t]$, then a polynomial $h(t) \in D[t]$ vanishes on x^M if and only if $h(t) \in D[t]f(t)$.*

Using Lemma 2.1, we get the following theorem which can be considered as some generalization of Wedderburn's Factorization theorem.

Theorem 2.2. *Let R be a ring, D a division subring of R and suppose M is a subgroup of R^* normalizing D^*. If $K = C_D(M)$ and $x \in D^*$ is algebraic over K with the minimal polynomial $f(t)$ of degree n, then there exist $x_1, \ldots, x_{n-1} \in x^{MD^*}$ such that*

$$f(t) = (t - x_{n-1}) \cdots (t - x_1)(t - x) \in D[t].$$

Proof. Take a factorization

$$f(t) = g(t)(t - x_r) \cdots (t - x_1)(t - x)$$

with $g(t) \in D[t]$, $x_1, \ldots, x_r \in x^{MD^*}$, where r is chosen as large as possible. We claim that $h(t) := (t - x_r) \cdots (t - x_1)(t - x)$ vanishes on x^M. In fact, consider an arbitrary element $y \in x^M$. If $h(y) \neq 0$, then, by [15, (16.3), p. 263], $g(x_{r+1}) = 0$, where $x_{r+1} = aya^{-1} \in x^{MD^*}, a = h(y)$. It follows that $g(t) = g_1(t)(t - x_{r+1})$ for some $g_1(t) \in D[t]$, and so

$$f(t) = g_1(t)(t - x_{r+1})(t - x_r) \cdots (t - x_1)(t - x).$$

Since this contradicts to the choice of r, we have $h(x^M) = 0$; so, in view of Lemma 2.1, $r = n - 1$. Hence, $f(t) = (t - x_{n-1}) \cdots (t - x_1)(t - x)$, as it was required to prove. □

We note that, the Wedderburn's Factorization theorem is the special case of Theorem 2.2, when we take $R = D$ and $M = D^*$.

From Theorem 2.2 and [15, (16.3), p. 263], we get the following corollaries.

Corollary 2.3. *Let R be a ring, D a division subring of R and suppose M is a subgroup of R^* normalizing D^*. Assume that $K = C_D(M)$ and $x \in D^*$ is algebraic over K with the minimal polynomial $f(t)$. If y is a root of $f(t)$ in D, then $y \in x^{MD^*}$.*

Corollary 2.4. *Let R be a ring, D a division subring of R and suppose M is a subgroup of R^* such that $D^* \trianglelefteq M$. Assume that $K = C_D(M)$ and $x \in D^*$ is algebraic over K with the minimal polynomial $f(t)$ of degree n. Then, K is contained in the center of D and there exists an element $c_x \in [M, x] \cap K(x)$ such that $x^n = N_{K(x)/K}(x)c_x$ with $N_{K(x)/K}(c_x) = 1$, where $N_{K(x)/K}$ is the norm of $K(x)$ to K.*

Proof. Since $D^* \leq M$, K is contained in the center of D and $K(x)$ is a field. Taking $b = N_{K(x)/K}(x)$, by Theorem 2.2 we have $b = x^{r_1} \cdots x^{r_n}$, with $r_1, \ldots, r_n \in M$. We can write b in the following form:

$$b = [r_1, x][r_2, x]^x[r_3, x]^{x^2} \cdots [r_n, x]^{x^{n-1}} x^n.$$

Putting

$$c_x^{-1} = [r_1, x][r_2^x, x][r_3^{x^2}, x] \cdots [r_n^{x^{n-1}}, x],$$

we have $c_x = b^{-1}x^n \in [M, x] \cap K(x)$. So,

$$N_{K(x)/K}(c_x) = N_{K(x)/K}(b^{-1})N_{K(x)/K}(x)^n = b^{-n}b^n = 1,$$

as it was required to prove. $\square$

This corollary can be reformulated in the following form which should be convenient in some cases of its application in the next sections.

Corollary 2.5. *Let R be a ring, D a division subring of R and suppose M is a subgroup of R^* normalizing D^*. Assume that $K = C_D(M)$ is a field and $x \in Z(D)^* \cap M$ is algebraic over K with the minimal polynomial $f(t)$ of degree n. Then, there exists an element $c_x \in [M, x] \cap K(x)$ such that $x^n = N_{K(x)/K}(x)c_x$ with $N_{K(x)/K}(c_x) = 1$.*

Proof. Let $b = N_{K(x)/K}(x) \in K$, by Theorem 2.2, we have $b = x^{r_1d_1} \cdots x^{r_nd_n}$, with $r_1, \dots, r_n \in M$ and $d_1, \dots, d_n \in D^*$. We can write b in the following form:

$$b = [r_1d_1, x][r_2d_2, x]^x[r_3d_3, x]^{x^2} \cdots [r_nd_n, x]^{x^{n-1}}x^n.$$

Since $x \in Z(D)^*$, we get

$$b = [r_1, x][r_2, x]^x[r_3, x]^{x^2} \cdots [r_n, x]^{x^{n-1}}x^n.$$

Putting

$$c_x^{-1} = [r_1, x][r_2^x, x][r_3^{x^2}, x] \cdots [r_n^{x^{n-1}}, x],$$

we have $c_x = b^{-1}x^n \in [M, x] \cap K(x)$. So,

$$N_{K(x)/K}(c_x) = N_{K(x)/K}(b^{-1})N_{K(x)/K}(x)^n = b^{-n}b^n = 1,$$

as it was required to prove. $\square$

In view of Corollary 2.4, the following fact is evident.

Corollary 2.6. *Let R be a ring, D a division subring of R and suppose M is a subgroup of R^* such that $D^* \trianglelefteq M$. If $K = C_D(M)$, then $K \subseteq Z(D)$ and the following assertions hold:*

- *(i)* *If D is algebraic over K, then $D^*/K^*[M, D^*]$ is a torsion group.*
- *(ii)* *If $[D : K] < \infty$, then $D^*/K^*[M, D^*]$ is a torsion group of a bounded exponent dividing $[D : K]$.*

For the illustration of the use of the generalization of Wedderburn's Factorization theorem, we are now ready to get the following result, which can be considered as a generalization of Theorem 1 in [19].

Theorem 2.7. *Let R be a ring, D a division subring of R and suppose M is a subgroup of R^* such that $D^* \trianglelefteq M$. If $K = C_D(M)$ and H/K is a finite separable field extension in D, then the following statements hold:*

- *(i)* *There exists an element $c \in [M, H^*]$ such that $N_{H/K}(c) = 1$ and $H = K(c)$.*
- *(ii)* *Either H is algebraic over the prime subfield $\mathbb{Z}_p$ or there exists an element $c \in [M, H^*]$ such that $N_{H/K}(c) = 1$ and $H = K(c) = K(c^2) = K(c^3) = \cdots$.*
- *(iii)* *If H has a nontrivial K-automorphism, then there exists an element $z = [m, h]$ with $m \in M$, $h \in H^*$ such that $N_{H/K}(z) = 1$ and $H = K(z)$.*

Proof. The proof of this theorem is similar to the proof of Theorem 1 in [19]. To get the proof of our theorem, it suffices to replace the use of Skolem-Noether theorem and Lemma A in the proof of Theorem 1 in [19] by the use of Corollary 2.3 and Corollary 2.4 respectively. $\square$

3 Maximal subgroups of subnormal subgroups

The main purpose in this section is to describe the structure of maximal subgroups in an arbitrary subnormal subgroup of D^* and show their influence to the whole structure of D. In the first, we prove the following useful lemmas we need for further study.

Lemma 3.1. *If D is a division ring with center F and G is a solvable-by-locally finite subnormal subgroup of D^*, then $G \subseteq F$.*

Proof. By assumption there is a solvable normal subgroup S of G such that G/S is locally finite. So, S is a solvable subnormal subgroup of D^* and in view of [23, 14.4.4, p. 440], we have $S \subseteq F$. Hence, $G/Z(G)$ is locally finite, so by [1, Lemma 3], G' is locally finite. Thus, G' is a torsion subnormal subgroup of D^* and by [12, Theorem 8], $G' \subseteq F$. Hence, G is solvable and by [23, 14.4.4, p. 440], $G \subseteq F$. ☐

Lemma 3.2. *Let D be a division ring with center F and G a subnormal subgroup of D^*. If G is locally polycyclic-by-finite (e.g. if G is locally nilpotent) then $G \subseteq F$.*

Proof. If $G \not\subseteq F$, then by Stuth's theorem (see, for example, [23, 14.3.8, p. 439]) we have $D = F(G)$. By a result of Lichtman [24, 4.5.2, p. 155] together with an exercise in [24, p. 162] or with the fact that G is contained in a unique maximal locally polycyclic-by-finite normal subgroup of D^*, it follows that G contains a noncyclic free subgroup, but this contradicts to the fact that G is locally polycyclic-by-finite. ☐

Lemma 3.3. *Let D be a division ring with center F, G a subnormal subgroup of D^* and suppose that M is a maximal subgroup of G. If $[M : Z(M)] < \infty$, then M is abelian.*

Proof. Suppose that M is non-abelian. Let $\{x_1, \ldots, x_t\}$ be a transversal of $Z(M)$ in M, E a subfield of D generated by $Z(M)$ and $K = E[M] = \{\sum_{i=1}^t a_i x_i | a_i \in E\}$. Then, K is a finite dimensional vector space over E and this implies that K is a division ring. Clearly, $E \subseteq Z(K)$, so we have $[K : Z(K)] < \infty$ and $C_K(M) = Z(K)$. Since M is maximal in G and M normalizes K^*, it follows that either G normalizes K^* or $K^* \cap G \subseteq M$.

Case 1: G normalizes K^*.

By Stuth's theorem, either $K \subseteq F$ or $K = D$. If $K \subseteq F$ then $M \subseteq F$, a contradiction. Thus, $D = K$; hence $Z(M) \subseteq C_D(M) = Z(D) = F$ and, consequently, $Z(M) = F^* \cap M$. Take an $x \in G \setminus M$ and put $H = \langle x, x_1, \ldots, x_t\rangle$. By maximality of M in G, we get $G = HZ(M) = H(F^* \cap M)$. It follows that $G' \leq H$, so H is normal in G. By [20, Theorem 1], we have $H \subseteq F$, hence $G \subseteq F$. In particular, M is abelian, a contradiction.

Case 2: $K^* \cap G \subseteq M$.

In this case $K^* \cap G = M$, so M is a subnormal subgroup of K^*. For $N = \langle x_1, \ldots, x_t\rangle$, we have $M = NZ(M)$, hence $M' \leq N$. Thus, N is a finitely generated subnormal subgroup of K^*. Recall that $[K : Z(K)] < \infty$, so in view of [20, Theorem 1], N is abelian and, consequently M is abelian, a contradiction. This completes our proof. ☐

It is well-known that the conjugacy class of a non-central element in D has the same cardinality with D (see, for example, [23, 14.2.2, p. 429]). This fact can be generalized by the following theorem which should be useful for the next study.

Theorem 3.4. *If D is a division ring with center F and G is a non-central subnormal subgroup of D^*, then $|x^G| = |D|$ for any $x \in D \setminus F$.*

Proof. We claim that $|x^G|$ is infinite for any $x \in D \setminus F$. Thus, assume that $|x^G| < \infty$, or, equivalently that $[G : C_G(x)] < \infty$. Since G normalizes $F(x^G)$, by Stuth's theorem it follows that $F(x^G) = D$. Putting $H = (C_G(x))_G$, we have $H \subseteq C_D(x^G) = C_D(D) = F$. On the other hand, by $[G : C_G(x)] < \infty$, $[G : H] < \infty$, so $G \subseteq F$ by Lemma 3.1, a contradiction. Therefore $|x^G|$ is infinite. Let R be a division ring generated by x^G, by Stuth's theorem, $R = D$, so $|x^G| = |D|$. $\square$

Using Theorem 3.4, in the following we show the role of an FC-element in maximal subgroups of a subnormal subgroup of D^*. The results obtained in this theorem will be used as principal tools in the proofs of next theorems.

Theorem 3.5. *Let D be a division ring with center F, G a subnormal subgroup of D^* and suppose that M is a maximal subgroup of G, containing a non-central FC-element α. By setting $K = F((\alpha)^M)$ and $H = C_D(K)$, the following conditions hold:*

(i) K *is a field,* $[K : F] = [D : H] < \infty$ *and* $F(M) = D$.

(ii) $K^* \cap G$ *is the FC-center and also the Fitting subgroup of* M.

(iii) K/F *is a Galois extension,* $M/H^* \cap G \cong \mathrm{Gal}(K/F)$ *is a finite simple group.*

(iv) *If* $H^* \cap G \subseteq K$, *then* $H = K$ *and* $[D : F] < \infty$.

Proof. First, we claim that $F(M) = D$. In fact, if $F(M)^* \cap G = M$, then M is a subnormal subgroup of $F(M)^*$ containing the non-central FC-element α, so we have a contradiction in view of Theorem 3.4. Thus, $F(M)^* \cap G = G$ by maximality of M in G; hence, by Stuth's theorem, we have $F(M) = D$.

Since α is an FC-element of M, $[M : C_M(\alpha)] < \infty$. Taking $N = (C_M(\alpha))_M$, $K = F(\alpha^M)$ and $H = C_D(K)$, we have $N \lhd M$, $N \leq H^*$ and $[M : N]$ is finite. Since M normalizes K^*, by maximality of M in G, it follows that, either G normalizes K^* or $K^* \cap G \leq M$. If G normalizes K^*, then by Stuth's theorem we have $K = D$, so $H = F$. Therefore, $N \leq F^* \cap M \leq Z(M)$. The inequality $[M : C_M(\alpha)] < \infty$ implies $[M : N] < \infty$ and consequently, $[M : Z(M)] < \infty$. Now, in view of Lemma 3.3, M is abelian, a contradiction. Thus $K^* \cap G = K^* \cap M \trianglelefteq M$, hence $K^* \cap G$ is a subnormal subgroup of K^* containing the set α^M of FC-elements in M. By Theorem 3.4, $\alpha^M \subseteq Z(K)$, hence K is a field. Since $H = C_D(K)$ and M normalizes K^*, M also normalizes H^*. By maximality of M in G, either G normalizes H^* or $H^* \cap G \leq M$. If G normalizes H^*, then by Stuth's theorem, either $H = D$ or $H \subseteq F$. However, both cases are impossible since $K \subseteq C_D(K) = H$ and K contains an element $\alpha \notin F = Z(D)$. Therefore, $H^* \cap G = H^* \cap M \trianglelefteq M$. Since $N \leq H^* \cap G$ and $[M : N] < \infty$, we have $[M : H^* \cap G] < \infty$ and $M = \bigcup_{i=1}^{t} x_i(H^* \cap G)$ for some transversal $\{x_1, \ldots, x_t\}$ of $H^* \cap G$ in M. Put $R = \sum_{i=1}^{t} x_i H$. Since M normalizes H^*, it is easy to see that R is a ring and also, it is a finite-dimensional right vector space over its division subring H. This fact implies that R is a division subring of D containing $F(M)$. Therefore $R = D$ and $[D : H] < \infty$. By using Double Centralizer theorem [15, 15.4, p. 253] we get $[K : F] = [D : H] < \infty$ and $Z(H) = C_D(H) = K$. Thus (i) is established.

Since M normalizes K^*, for all $a \in M$ the mapping $\theta_a : K \to K$ given by $\theta_a(x) = axa^{-1}$ is an automorphism of K/F. Now, consider the mapping $\psi : M \to \mathrm{Gal}(K/F)$ given by $\psi(a) = \theta_a$. Clearly, ψ is a group homomorphism with $\ker\psi = C_M(K^*) = C_D(K)^* \cap M = H^* \cap M = H^* \cap G$. Since $F(M) = D$, we obtain $C_D(M) = F$ and it follows that the fixed field of $\psi(M)$ is F. By Galois correspondence, we conclude that ψ is a surjective homomorphism and K/F is a Galois extension. Hence, $M/H^* \cap G \cong \mathrm{Gal}(K/F)$ is a finite group. Assume that $\mathrm{Gal}(K/F)$ is not simple. Then, there exists some intermediate subfield $L, F \subset L \subset K$, such that $\theta(L) = L$ for all $\theta \in \mathrm{Gal}(K/F)$. Thus, L and $E = C_D(L)$ are normalized by M and clearly, $E \neq D$ and $E \not\subseteq F$. Therefore, by

Stuth's theorem, E^* is not normalized by G. If $E^* \cap G \not\subseteq M$, then $G = M(E^* \cap G)$ normalizes E^*, a contradiction. Hence, $E^* \cap G \leq M$. Since $K^* \cap G \trianglelefteq M$ and $K^* \cap G \leq E^* \cap G$, we have $K^* \cap G \trianglelefteq E^* \cap G$ and so $K^* \cap G$ is an abelian subnormal subgroup of E^*. By Lemma 3.1, we have $K^* \cap G \subseteq Z(E) = L$. Hence, $K = F(\alpha^M) \subseteq F(K^* \cap G) \subseteq L$, a contradiction. Thus $\mathrm{Gal}(K/F)$ is simple, and the proof of (iii) is complete.

If $H^* \cap G \subseteq K$, then $H^* \cap G = K^* \cap G$. Recall that $[M : H^* \cap G] < \infty$, hence $[M : K^* \cap G] < \infty$. Suppose that $\{y_1, \ldots, y_l\}$ is a transversal of $K^* \cap G$ in M. Since M normalizes K^*, $Q = \sum_{i=1}^{l} y_i K$ is a division ring. Clearly, Q contains both F and M, so in view of (i), we have $Q = D$. It is easy to see that $[D : K] = [Q : K] \leq |M/H^* \cap G| = |\mathrm{Gal}(K/F)| = [K : F]$. Hence, by Double Centralizer theorem, K is a maximal subfield of D, $K = H$ and $[D : F] < \infty$. Thus (iv) is established.

To prove (ii), first we claim that $K^* \cap G$ is a maximal abelian normal subgroup of M. Thus, suppose C is a maximal abelian normal subgroup of M containing $K^* \cap G$. Then, $\alpha^M \subseteq C$ and it follows $C \leq H^*$. Recall that $H^* \cap G \leq M$, so we have $C \trianglelefteq H^* \cap G$. Hence, C is an abelian subnormal subgroup of H^*, and by Lemma 3.1, $C \subseteq Z(H) = K$. So $K^* \cap G = C$ is a maximal abelian normal subgroup of M.

To prove $K^* \cap G$ is the Fitting subgroup of M, it suffices to show that $K^* \cap G$ is a maximal nilpotent normal subgroup of M. Now, assume that A is a nilpotent normal subgroup of M which strictly contains $K^* \cap G$. Then, $B = H^* \cap G \cap A$ is a nilpotent subnormal subgroup of H^*. Hence, by Lemma 3.1, we conclude that $B \subseteq Z(H) = K$ and consequently, $B = K^* \cap G \cap A$. If $A \subseteq H^* \cap G$, then $A = B \subseteq K^* \cap G$, a contradiction. Therefore, $A \not\subseteq H^* \cap G$. Thus, $A(H^* \cap G)/H^* \cap G$ is a nontrivial normal subgroup of $M/H^* \cap G$. Since $M/H^* \cap G$ is simple, $M/H^* \cap G = A(H^* \cap G)/H^* \cap G \cong A/B$. Suppose $S = \sum_{i=1}^{m} z_i K$, where $\{z_1, \ldots, z_m\}$ is a transversal of B in A. Since A normalizes K^* and $B \subseteq K$, S is a division ring and $[S : K]_r \leq m$. Recall that M normalizes A and K; so, M also normalizes S. By maximality of M in G, it follows either G normalizes S or $S^* \cap G \leq M$. If the second case occurs, then A is a nilpotent subnormal subgroup of S^*. By Lemma 3.1, A is abelian and this contradicts to the fact that $K^* \cap G$ is a maximal abelian normal subgroup of M. Thus, G normalizes S, and by Stuth's theorem we have $S = D$. Therefore, $[D : K]_r \leq m = |M/H^* \cap G| = |\mathrm{Gal}(K/F)| = [K : F]$. This implies that $[D : F] = m^2$ and $K = H$ is a maximal subfield of D. From $M/H^* \cap G = A(H^* \cap G)/H^* \cap G$ and $K^* \cap G < A$, it follows $M = A$. Since $[D : F] < \infty$, M can be considered as a nilpotent linear group no containing unipotent elements ($\neq 1$). By a result in [3, p. 114], $[M : Z(M)]$ is finite, which contradicts to Lemma 3.3. Thus, $K^* \cap G$ is the Fitting subgroup of M.

For any $x \in K^* \cap G$, the elements of $x^M \subseteq K$ have the same minimal polynomial over F, so $|x^M| < \infty$. For any $x \in M \setminus K^* \cap G$, if $|x^M| < \infty$ then, by what we proved, $F(x^M) \cap G$ is the Fitting subgroup of M which is different from $K^* \cap G$, a contradiction. Hence, $|x^M| = \infty$ and (ii) follows. Thus, the proof of the theorem is now complete. □

From Theorem 3.5 we get the following corollary which is more convenient for further applications.

Corollary 3.6. *Let D be a division ring with center F, G a subnormal subgroup of D^* and suppose M is a maximal subgroup of G. If M contains an abelian normal subgroup A and an element $\alpha \in A \setminus Z(M)$ algebraic over $F(Z(M))$, then $K = F(A)$ and $H = C_D(K)$ satisfy (i) - (iv) in Theorem 3.5. Moreover, $F(A) = F[A]$.*

Proof. Since $A \trianglelefteq M$, the elements of α^M in the field $F(Z(M)A)$ have the same minimal polynomial over $F(Z(M))$. Hence, $|\alpha^M| < \infty$. Now, by using Theorem 3.5, $K = F(\alpha^M)$ and $H = C_D(K)$ satisfy (i) - (iv) in Theorem 3.5. By (ii), $A \subseteq K$ and $K = F(A) = F[A]$ since $[K : F] < \infty$. □

Theorem 3.7. *Let D be a division ring with center F, G a subnormal subgroup of D^* and suppose M is a non-abelian metabelian maximal subgroup of G. Then $[D:F]=p^2$, $M/M'Z(M)\cong\bigoplus_{i\in I}\mathbb{Z}_p$, where p is a prime number, and there exists a maximal subfield K of D such that K/F is a Galois extension, $\mathrm{Gal}(K/F)\cong M/K^*\cap G\cong\mathbb{Z}_p$ and $K^*\cap G$ is the FC-center and also the Fitting subgroup of M. Furthermore, for any $x\in M\setminus K$ we have $x^p\in F$ and $D=F[M]=\bigoplus_{i=1}^{p}Kx^i$.*

Proof. Denote by A a maximal abelian normal subgroup of M containing M' (we note that such a subgroup exists since M' is abelian). Consider an arbitrary subgroup N of M which properly contains A. Since $M'\le N$, $N\trianglelefteq M$; so the maximality of M in G implies that either G normalizes $F(N)^*$ or $F(N)^*\cap G\le M$. If the second case occurs, then $N\trianglelefteq F(N)^*\cap G$, so N is a metabelian subnormal subgroup of $F(N)^*$. By Lemma 3.1, N is abelian, which contradicts to the maximality of A. Hence, G normalizes $F(N)^*$ and by Stuth's theorem we conclude that $F(N)=D$.

Now let a be an element from $M\setminus A$ and assume that a is transcendental over $F(A)$. Put $T=F(A)^*\langle a^2\rangle$; since a normalizes $F(A)^*$, it is not hard to see that $F[T]=\bigoplus_{i\in\mathbb{Z}}F(A)a^{2i}$ and $(F[T],F(A),T,T/F(A)^*)$ is a crossed product. But $T/F(A)^*\cong\langle a^2\rangle$ is an infinite cyclic group and by [[**?**], 1.4.3, p. 26], $F[T]$ is an Ore domain. On the other hand, by what we proved before, we have $F(T)=D$. Therefore, there exist two elements $s_1,s_2\in F[T]$ such that $a=s_1s_2^{-1}$. Writing $s_1=\sum_{i=l}^{m}k_ia^{2i}$ and $s_2=\sum_{i=l}^{m}k_i'a^{2i}$, with $k_i,\ k_i'\in F(A)$, for any $l\le i\le m$, we have $\sum_{i=l}^{m}ak_i'a^{2i}=\sum_{i=l}^{m}k_ia^{2i}$. If we set $l_i=ak_i'a^{-1}$, for any $l\le i\le m$, then l_i's are elements of $F(A)$ and we have $\sum_{i=l}^{m}l_ia^{2i+1}=\sum_{i=l}^{m}k_i'a^{2i}$. This shows that a is algebraic over $F(A)$, say of degree n. Using the fact that a normalizes $F(A)^*$, we see that $R=\bigoplus_{i=0}^{n-1}F(A)a^i$ is a domain, which is a finite-dimensional left vector space over $F(A)$. Therefore R is a division ring and $R=F(A\langle a\rangle)$. By what we proved before we conclude that $R=D$, hence $[D:F(A)]_l<\infty$. So, by Double Centralizer theorem, $[D:F]<\infty$.

If $A\le Z(M)$, then, since $M'\le A$, $\langle A,x\rangle$ is an abelian normal subgroup of M properly containing A, for any $x\in M\setminus A$. But, this contradicts to the maximality of A; hence $A\not\subseteq Z(M)$. Since $[D:F]<\infty$, all elements of $A\setminus Z(M)$ is algebraic over F. In view of Corollary 3.6, there exists a subfield K of D such that K and $H=C_D(K)$ satisfy $(i)-(iv)$ in Theorem 3.5. So, $H^*\cap G$ is a metabelian subnormal subgroup of H^*. By Lemma 3.1, $H^*\cap G\subseteq Z(H)=K$. Thus, (iv) implies that $K=H$ is a maximal subfield of D. Since M is metabelian, $M/K^*\cap G$ is simple and metabelian. We conclude that $\mathrm{Gal}(K/F)\cong M/K^*\cap G\cong\mathbb{Z}_p$, where p is a prime number and so $[D:F]=p^2$. Since $[M:K^*\cap G]=p$ and D is algebraic over F, for any $x\in M\setminus K$, we have $D=F(M)=F[M]=\sum_{i=1}^{p}Kx^i$ and, since $[D:K]=p$, $D=\bigoplus_{i=1}^{p}Kx^i$. Suppose $x^p\notin F$. By $F(M)=D$, we have $Z(M)=M\cap F^*$, and it follows $C_M(x^p)\neq M$. On the other hand, we note that $\langle x,K^*\cap G\rangle\le C_M(x^p)$ and $[M:K^*\cap G]$ is prime, so we get $C_M(x^p)=M$, a contradiction. Thus, $x^p\in F$. Now, by Corollary 2.5, for any $y\in K^*\cap M=K^*\cap G$, we have $y^p\in M'F^*$. Therefore $y^p\in M'(M\cap F^*)=M'Z(M)$, so $M/M'Z(M)$ is an abelian group of exponent p and by Bear-Prufer's theorem [22, p. 105], $M/M'Z(M)\cong\bigoplus_{i\in I}\mathbb{Z}_p$. This completes the proof. □

In [4], Ebrahimian proved that if M is a nilpotent maximal subgroup of D^*, then M is abelian. The following theorem shows that this remains also true if we replace D^* by any its subnormal subgroup.

Theorem 3.8. *Let D be a division ring and G a subnormal subgroup of D^*. If M is a nilpotent maximal subgroup of G, then M is abelian.*

Proof. Assume that M is non-abelian. Then, there exists an element $x\in Z_2(M)\setminus Z(M)$. By considering the homomorphism $f:M\to Z(M)$ given by $f(y)=xyx^{-1}y^{-1}$ we see that $M/C_M(x)$

is an abelian group, so $M' \leq C_M(x)$. Put $F = Z(D)$; since $x \notin Z(M)$, we have $F(M') \neq D$. Now we claim that M' is abelian. If G normalizes $F(M')$, then, by Stuth's theorem, we have $F(M') \subseteq F$, so M' is abelian, as claimed. Now, suppose that G does not normalize $F(M')$. Since M normalizes $F(M')$ and M is maximal in G, it follows that $F(M') \cap G \subseteq M$. Thus, $M' \trianglelefteq F(M') \cap G$ is a nilpotent subnormal subgroup of $F(M')^*$ and so, by Lemma 3.1, M' is abelian, as claimed. Therefore M is metabelian. By Theorem 3.7, we conclude that the Fitting subgroup of M is a proper subgroup of M, a contradiction. Thus, M is abelian as it was required to prove. $\square$

Lemma 3.9. *Let D be a centrally finite division ring with center F. Then, $D' \cap F^*$ is finite.*

Proof. Suppose $[D : F] = n^2$. By taking the reduced norm, we obtain $x^n = 1$ for all $x \in D' \cap F^*$. Since F is a field, $D' \cap F^*$ is finite. $\square$

Lemma 3.10. *Let D be a division ring with center F, G a subnormal subgroup of D^* and suppose M is a non-abelian maximal subgroup of G. If M' is locally finite, then $M/Z(M)$ is locally finite, M' is locally cyclic and the conclusions of Theorem 3.7 follow.*

Proof. Suppose M' is non-abelian, so we need only to consider the case Char$D$=0.

First, we claim that there exists a torsion abelian normal subgroup of M which is not contained in $Z(M)$. By maximality of M in G, either $F(M') \cap G \subseteq M$ or G normalizes $F(M')$. If $F(M') \cap G \subseteq M$, then M' is a locally finite subnormal subgroup of $F(M')^*$, so M' is abelian by Lemma 3.1, a contradiction. So, G normalizes $F(M')$ and by Stuth's theorem we have $F[M'] = F(M') = D$. On the other hand, by [24, 2.5.5, p. 74], there exists a metabelian normal subgroup of M' of finite index n. Let $Q = \langle \{x^n | x \in M'\} \rangle$. Then, Q is a metabelian normal subgroup of M and Q' is a torsion abelian normal subgroup of M. If Q' is not contained in $Z(M)$ then we are done. Hence, we may assume that $Q' \leq Z(M)$, so Q is locally finite and nilpotent. Thus, by [24, 2.5.2, p. 73], Q contains an abelian subgroup B such that $[Q : B] < \infty$. Clearly, $C = \cap_{x \in M} xBx^{-1}$ is a torsion abelian normal subgroup of M and we may assume that $C \leq Z(M)$. On the other hand, $[Q : B] < \infty$ and $Q \trianglelefteq M$, so it follows that Q/C has bounded exponent. From definition of Q and $C \leq Z(M)$, $M'/Z(M')$ has also bounded exponent. Therefore, by the classification theorem of locally finite groups in division ring [24, 2.5.9, p. 75], we conclude that M' is abelian-by-finite. Now, let A be an abelian normal subgroup of M' of finite index. Since $F[M'] = D$, D is finite-dimensional over a field $F[A]$. The Double Centralizer theorem implies that $[D : F] < \infty$ and by Lemma 3.9, $|A \cap F^*| < \infty$. Also by $F[M'] = D$, we have $Z(M') = F \cap M'$ and hence, $A/A \cap F^*$ has bounded exponent, so is A. Since A can be considered as the multiplicative subgroup of a field, we conclude that A is finite, so is M'. Hence, M is an FC-group and so it is abelian by Theorem 3.5, a contradiction. Therefore, there exists a torsion abelian normal subgroup of M which is not contained in $Z(M)$.

Now, by Corollary 3.6, there exists a field K such that K and $H = C_D(K)$ satisfy $(i) - (iv)$ in Theorem 3.5. Taking $N = M' \cap H^*$, we have $N \trianglelefteq M \cap H^* = G \cap H^*$, so N is a locally finite subnormal subgroup of H^*. By Lemma 3.1, $N = M' \cap H^*$ is abelian and so $M' \nsubseteq M \cap H^*$. If $M/M \cap H^*$ is abelian, then $M' \subseteq M \cap H^*$, a contradiction. Thus, from (iii) in Theorem 3.5, $M'/N \cong M'(M \cap H)/M \cap H = M/M \cap H$ is a non-abelian finite simple group. Let $\{x_1, \ldots, x_m\}$ be a transversal of N in M' and $S = \langle x_1, \ldots, x_m \rangle$ a finite subgroup of M'. Since S has a quotient group which is a finite simple non-abelian group, S is unsolvable. The classification theorem of finite groups in division ring [24, 2.1.4, p. 46] states that the only unsolvable finite subgroup of a division ring is $SL(2, 5)$, so $M' = S \cong SL(2, 5)$ is a finite group. Thus, M is an FC-group and so

it is abelian by Theorem 3.5, a contradiction. Therefore, M' is abelian and it can be considered as a torsion multiplicative subgroup of a field. Hence M' is locally cyclic and Theorem 3.7 completes the proof. $\square$

4 The existence of noncyclic free subgroups

The question on the existence of noncyclic free subgroups in the multiplicative group of a division ring was posed by Lichtman in [16]. In [5], Goncalves proved that if D is centrally finite division ring (i.e. D is finite dimensional vector space over its center F), then D^* contains a noncyclic free group. Later, in [7], Goncalves and Mandel posed the more general question: is it true that every subnormal subgroup of the multiplicative group of a division ring contains a noncyclic free subgroup? In this work, the authors show that this is true in some particular cases. More exactly, they proved that a subnormal subgroup G of D^* contains a noncyclic free subgroup if G contains some element $x \in D \setminus F$, which is algebraic over the center F of D such that either (a) $\mathrm{Gal}(F(x)/F) \neq 1$ or (b) $x^p \in F$, where $p = 2$ or $p = \mathrm{char}(F) > 0$. The affirmative answer was also obtained for centrally finite division rings by Goncalves in his precedent work [6]. Note that the affirmative answer to the question above would imply some known theorems like the commutativity theorems of Kaplansky, Jacobson, Hua, Herstein, Stuth,... For more information we refer to [11]. In this section we shall prove that the question above has the affirmative answer for locally finite division rings. Recall that a division ring D with the center F is *locally finite* if for every finite subset S of D, the division subring $F(S)$ is finite dimensional vector space over F. It can be prove that in a locally finite division ring every finite subset generates the centrally finite division subring. From this observation, the following definition seems to be quite natural: a division ring D is called *weakly locally finite* if for every finite subset S of D, the division subring of D generated by S is centrally finite. This notion is introduced firstly in [9]. Clearly, every locally finite is weakly locally finite. The converse is not true. In [9], using the well-known method due to Mal'cev and von Neumann, the authors constructed the example of weakly locally finite division ring which is not even algebraic. In the next two lemmas we prove some facts for weakly locally finite division rings.

Lemma 4.1. *Let D be a weakly locally finite division ring of characteristic 0 and G be a subgroup of D^*. If G contains no noncyclic free subgroup, then G is locally solvable -by-finite.*

Proof. Take an arbitrary finite subset S of G, and let K be a division subring of D generated by S. Then, K is centrally finite , so we have $m = [K : Z(K)] < \infty$. Therefore, the subgroup $< S >$ generated by S in G can be considered as a subgroup of $\mathrm{GL}_m(Z(K))$. By [25, Theorem 1], there exists a solvable subgroup H of $< S >$ of finite index. If $N = H_G$, then N is a solvable normal subgroup of finite index in $< S >$. Hence, G is locally solvable-by-finite. $\square$

Lemma 4.2. *Let D be a weakly locally finite division ring and G be a subgroup of D^*. If G contains no noncyclic free subgroup, then G is (locally solvable) -by-(locally finite).*

Proof. Firstly, we claim that G is locally solvable-by-locally finite. If $\mathrm{char}(D) = 0$, then this claim follows from Lemma 4.1. Assume that $\mathrm{char}(D) = p > 0$. Then, by the same argument as in the proof of Lemma 4.1, for any finite subset S of G, the subgroup $< S >$ generated by S can be viewed as a linear group over the field of characteristic $p > 0$. By [25, Theorem 2], $< S >$ contains a solvable normal subgroup H such that $< S > /H$ is locally finite. Hence, G is locally solvable-by-locally finite. Now, by [24, 3.3.9, p. 103], G is (locally solvable) -by-(locally finite). $\square$

Lemma 4.3. *Let D be a locally finite division ring and G an arbitrary subgroup of D^*. Then either G contains a noncyclic free subgroup or G is abelian-by-locally finite.*

Proof. Assume that G does not contain any noncyclic free subgroups. By Lemma 4.2, G is (locally solvable)-by-(locally finite). The Wehrfritz's theorem [24, 5.5.1, p. 199] implies that there exists an abelian normal subgroup A of G such that G/A is locally finite. Therefore, G is abelian-by-locally finite. □

From Lemma 3.1 and Lemma 4.3 we get the following result, which shows that Conjecture 2 in [7] is true for locally finite division rings.

Theorem 4.4. *Let D be a locally finite division ring with center F and G a subnormal subgroup of D^*. If G is noncentral, then G contains a noncyclic free subgroup.*

The next theorem can be considered as a broad generalization of the main theorem in [21].

Theorem 4.5. *Let D be a locally finite division ring with center F, G a subnormal subgroup of D^* and suppose M is a non-abelian maximal subgroup of G. If M contains no noncyclic free subgroups, then $[D : F] < \infty$, $F(M) = D$ and there exists a maximal subfield K of D such that K/F is a Galois extension, $\mathrm{Gal}(K/F) \cong M/K^* \cap G$ is a finite simple group and $K^* \cap G$ is the FC-center and also the Fitting subgroup of M.*

Proof. First, we show that $F(M) = D$. If $F(M) \neq D$, then, by Stuth's theorem, $F(M)$ is not normalized by G. So, by maximality of M in G, we have $F(M) \cap G = M$. Thus, M is a non-abelian subnormal subgroup of $F(M)^*$ no containing a noncyclic free subgroup, which contradicts to Theorem 4.4. So, we have $F(M) = D$; hence $Z(M) = M \cap F^*$. Now, by Lemma 4.3, there exists an abelian normal subgroup A of M such that M/A is locally finite. If $A \subseteq Z(M)$, then $M/Z(M)$ is locally finite, so is M' and the result follows from Lemma 3.10. Assume that $A \not\subseteq Z(M)$. By applying Corollary 3.6, there exists a subfield K of D such that $H = C_D(K)$ and K satisfy $(i) - (iv)$ in Theorem 3.5. Since $H^* \cap G$ is a subgroup of M and also a subnormal subgroup of H^*, by Theorem 4.4 we have $H^* \cap G \subseteq K$. Now, the condition (iv) in Theorem 3.5 completes our proof. □

5 Maximal subgroups of $GL_1(D)$

In this section we investigate the similar problem as in Section 3 for the particular case, when $G = D^*$.

Theorem 5.1. *Let D be a division ring with center F and M a maximal subgroup of D^*, containing a non-central FC-element α. By setting $K = F(\alpha^M)$, the following conditions hold:*

- *(i)* K *is a field,* $[K : F] < \infty$ *and* $F[M] = D$.
- *(ii)* K^* *is the Hirsch-Plotkin radical of* M.
- *(iii)* $F^* < K^* \leq C_D(K)^* < M < D^*$.
- *(iv)* K/F *is a Galois extension,* $M/C_D(K)^* \cong \mathrm{Gal}(K/F)$ *is a finite simple group.*
- *(v)* *For any* $x \in K$, $|x^M| \leq [K : F]$ *and for any* $y \in D \setminus K$, $|y^M| = |D|$.

Proof. We note that K and $H = C_D(K)$ satisfy $(i) - (iii)$ in Theorem 3.5 (for $G = D^*$). As in first paragraph of the proof of Theorem 3.5, we can write $D = \sum_{i=1}^{t} x_i H$, where $x_1, \ldots, x_t \in M$. In this case, we have $H^* \leq M$, and this implies $F[M] = D$. It remains to prove (ii) and (v). Let

x be an element of K^*. Since K is algebraic over F and $K^* \lhd M$, $x^M \subseteq K$ and the elements of x^M have the same minimal polynomial f over F. So $|x^M| \leq \deg(f) \leq [K : F]$. For any $y \in D \setminus K$, $C_H(y)$ is a proper division subring of H since $y \notin K = C_D(H)$. By [23, 14.2.1, p. 429] and $[D : H] < \infty$ we have $|y^M| = [M : C_M(y)] \geq [H^* : C_H(y)^*] = |H| = |D|$. Thus, $(i), (iii), (iv)$ and (v) hold.

Now we prove (ii). Let A be the Hirsch-Plotkin radical of M and suppose $K^* < A$. Then, $B = H^* \cap A$ is a locally nilpotent normal subgroup of H^*. Hence, by Lemma 3.2, we conclude that $B \subseteq Z(H) = K$ and consequently, $B = K^*$. If $A \subseteq H^*$, then $A = B = K^*$, a contradiction. Therefore, $A \nsubseteq H^*$. Thus, AH^*/H^* is a nontrivial normal subgroup of M/H^*. Since M/H^* is simple, $M/H^* = AH^*/H^* \cong A/B$. Suppose $R = \sum_{i=1}^m y_i K$, where $\{y_1, \ldots, y_m\}$ is a transversal of B in A. Since A normalizes K^* and $B \subseteq K$, R is a division ring and $[R : K]_r \leq m$. Since R is a division ring generated by A and K, M normalizes R; so by maximality of M in D^*, it follows that either D^* normalizes R or $R^* \leq M$. If the second case occurs, then A is a locally nilpotent normal subgroups of R^*. By Lemma 3.2, A is abelian and this contradicts to the fact that K^* is the Fitting subgroup of M (Theorem 3.5 (ii)). Thus, D^* normalizes R, and by Stuth's theorem we have $R = D$. Therefore, $[D : K]_r \leq m = |M/H^*| = |\mathrm{Gal}(K/F)| = [K : F]$. This implies that $[D : F] = m^2$ and $K = H$ is a maximal subfield of D. From $M/H^* = AH^*/H^*$ and $K^* < A$, it follows $M = A$. Since $F[M] = D$, M is a locally nilpotent absolutely irreducible subgroup of D^* and by [24, 5.7.11 p. 215], $M/Z(M)$ is locally finite. Thus, K/F is a nontrivial radical Galois extension and by [15, 15.13, p. 258], F is algebraic over a finite field, so is D. Therefore, by Jacobson's theorem [15, p. 219], D is a field, a contradiction. Thus, $K^* = A$ is the Hirsch-Plotkin radical of M. □

The proof of the following corollary is similar as the proof of Corollary 3.6.

Corollary 5.2. *Let D be a division ring with center F and suppose M is a maximal subgroup of D^*. Assume that M contains an abelian normal subgroup A and there exists some element $\alpha \in A \setminus Z(M)$ such that α is algebraic over $F(Z(M))$. Then, $K = F[A]$ satisfies (i) - (v) in Theorem 5.1.*

Lemma 5.3. *Let D be a division ring with center F and suppose M is a maximal subgroup of D^* such that $C_D(M) = F$ and $F[M]^* = M$. If M' is algebraic over F, then $F[M']$ is an algebraic division F-algebra and $F[M']^* \trianglelefteq M$.*

Proof. First, we claim that if $x \in M$ and $g(t) = (t-r_n)\cdots(t-r_1) \in D[t]$ with $r_n, \ldots, r_1 \in x^M$, then $g(x) \in M \cup \{0\}$. Let $h(t) = (t-r_{n-1})\cdots(t-r_1)$, by induction we have $h(x) \in M \cup \{0\}$. If $h(x) = 0$, then $g(x) = 0$ as claimed. If $h(x) \neq 0$, then, by [15, (16.3), p. 263], we have $g(x) = (x^{h(x)} - r_n)h(x)$. Take $r_n = x^m$ with $m \in M$, we have $x^{h(x)} - r_n = (x^{m^{-1}h(x)} - x)^m = ([m^{-1}h(x), x] - 1)^m x^m$. On the other hand, $[m^{-1}h(x), x] - 1 \in M$ since M' is algebraic over F and $F[M]^* = M$. Thus, $x^{h(x)} - r_n \in M$ and so $g(x) \in M$.

Next, by the same argument as in the proof of Theorem 2.2 and what we proved, if $x \in M$ is algebraic over F with minimal polynomial $f(t)$ of degree n, then there exists $x_1, \ldots, x_{n-1} \in x^M$ such that

$$f(t) = (t - x_{n-1})\cdots(t - x_1)(t - x) \in D[t].$$

Also by the same argument as in the proof of Corollary 2.4, we have $x^n \in M'F^*$. Thus, if $x, y \in M$ are algebraic over F, then $xM'F^*$ and $yM'F^*$ are two elements of $M/M'F^*$ of finite order and so $xyM'F^*$ is of finite order. Since M' is algebraic over F, so is xy. If $x, y \in M$ are algebraic over F, then, by $F[M]^* = M$, $x + y = x(1 + x^{-1}y)$ is algebraic over F and belongs to M. Thus, $F[M']$ is an algebraic division F-algebra and $F[M']^* \subseteq F[M]^* = M$. This completes our proof. □

In the next theorem we get some result as in [1, Theorem 6], but with a weaker condition. In fact, we replace the condition of algebraicity of M by the condition of algebraicity of derived subgroup M'.

Theorem 5.4. *Let D be a division ring with center F and M a non-abelian locally solvable maximal subgroup of D^* with M' is algebraic over F. Then, $[D : F] = p^2$, $M/M'F^* \cong \bigoplus_{i\in I} \mathbb{Z}_p$, where p is a prime number, and there exists a maximal subfield K of D such that K/F is a Galois extension, $\mathrm{Gal}(K/F) \cong M/K^* \cong \mathbb{Z}_p$ and K^* is the FC-center and also the Hirsch-Plotkin radical of M. Furthermore, for any $x \in M \setminus K^*$ we have $x^p \in F$, $M = \bigcup_{i=1}^p K^*x^i$ and $D = F[M] = \bigoplus_{i=1}^p Kx^i$.*

Proof. If $F(M) \neq D$, then by maximality of M, $M \cup \{0\} = F(M)$ is a division ring. By Remark 2 in [4], M is abelian, a contradiction. Hence $F(M) = D$ and $C_D(M) = F$. Suppose that $F[M] \neq D$, so $F[M]^* = M$ by maximality of M. By Lemma 5.3, $F[M']$ is a division ring whose multiplicative group is locally solvable. The Remark 2 in [4] implies that M' is abelian, so by Theorem 3.7, we have $F[M] = D$, a contradiction. Therefore, M is absolutely irreducible and by Corollary 4 of [27], M is abelian-by-locally finite. Thus, there exists an abelian normal subgroup A of M such that M/A is locally finite. We need to show that D is locally finite over F. Let us examine two possible cases.

Case 1: $A \cap M' \subseteq F$

We note that $F(M')^*$ is normalized by M. By maximality of M, either $F(M') \cap G \subseteq M$ or G normalizes $F(M')^*$. In the first case, M' is a subnormal subgroup of $F(M')^*$. On the other hand, M is abelian-by-locally finite, so is M'. By Lemma 3.1, M' is abelian; hence, by Theorem 3.7, $[D : F] < \infty$, as claimed. In the second case, G normalizes $F(M')^*$ and by Stuth's theorem, either $F(M') \subseteq F$ or $F(M') = D$. If $F(M') \subseteq F$, then, by Theorem 3.7, $[D : F] < \infty$ and we are done. Suppose that $F(M') = D$. We know that $M'/M' \cap A \cong AM'/A \leq M/A$ is locally finite, so $M'/M' \cap F$ is locally finite since $M' \cap A \subseteq F$. Therefore, $D = F(M')$ is locally finite over F.

Case 2: $A \cap M' \not\subseteq F$

Since M' is algebraic over F, there exists an element $x \in A \cap M' \setminus F$ algebraic over F. By Corollary 5.2, there exists a subfield K_1 of D such that K_1 satisfies $(i) - (v)$ in Theorem 5.1. So, $C_D(K_1)^* \leq M$ is abelian-by-locally finite and by Lemma 3.1, $C_D(K_1)^*$ is abelian. Now, by Double Centralizer theorem, $[D : F] < \infty$. Therefore, D is locally finite. Since M is locally solvable, M contains no noncyclic free subgroups. By Theorem 4.5, there exists a maximal subfield K of D as in Theorem 4.5. Since M is locally solvable and M/K^* is a finite simple group, we have $M/K^* \cong \mathbb{Z}_p$, where p is a prime number. Thus, $M' \leq K^*$ and M is metabelian. Now, the proof is complete by applying of Theorems 3.7 and 5.1. □

In [8, Theorem 3.2], it was proved that, if D is a non-commutative division ring, algebraic over its center, then any locally nilpotent maximal subgroup of D^* is the multiplicative group of some maximal subfield of D. Here, we are now ready to generalize this result by the following theorem.

Theorem 5.5. *Let D be a non-commutative division ring with center F and M a locally nilpotent maximal subgroup of D^*. If M' is algebraic over F, then M is abelian. Consequently, M is the multiplicative group of some maximal subfield of D.*

Proof. If M is non-abelian. Using Theorem 5.4, we conclude that the Hirsch-Plotkin radical of M is a properly subgroup of M, this contradicts to the fact that M is locally nilpotent. Thus M is abelian. The last conclusion is evident. □

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